RADIAL SYMMETRY AND MONOTONICITY RESULTS FOR AN INTEGRAL EQUATION

LI MA AND DEZHONG CHEN

Abstract. In this paper, we consider radial symmetry property of positive solutions of an integral equation arising from some higher order semi-linear elliptic equations on the whole space $\mathbf{R}^n$. We do not use the usual way to get symmetric result by using moving plane method. The nice thing in our argument is that we only need a Hardy-Littlewood-Sobolev type inequality. Our main result is Theorem 1 below.

1. Introduction

In the study of standing waves of the non-linear Klein-Gordon equations we meet the following semi-linear elliptic equation (EE):

$$-\Delta u + u = u|u|^{\beta-1}, \quad on \quad \mathbf{R}^n$$

where $\beta > 1$ is a positive constant. This equation also appears from the ground states of the Schrodinger equation [2]. It is shown that smooth positive solutions of (EE) are unique and radial symmetric, so it has a nice decay at infinity [7]. In this paper, we consider radial symmetry property of positive solutions of an integral equation arising from some higher order semi-linear elliptic equations on the whole space $\mathbf{R}^n$. Our work is motivated from the works of Chen, Li, and Ou [4] [5], where the authors studied the Yamabe type equations. The usual way to get symmetric result is using the classical moving plane method. However, the moving plane method is limited since it uses the Maximum principle. The nice thing observed by Chen-Li-Ou [4] is that one only needs a Hardy-Littlewood-Sobolev type inequality to get symmetric result for positive solutions of elliptic equations of Yamabe type.

For $\alpha > 0$, $\beta > 1$, we consider positive solutions for the following semi-linear partial differential equation in $\mathbf{R}^n$:

$$(I - \Delta)^{\alpha/2}(u) = u^\beta \tag{1}$$

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where $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$ denotes the Laplace operator in $\mathbf{R}^n$. With some decay assumption of solutions at infinity, it is well-known that (1) is equivalent to the following integral equation

$$u = g_\alpha * u^\beta, \tag{2}$$

where $*$ denotes the convolution, and g_α is the Bessel kernel (for its precise definition, see §2).

Our main result is

Theorem 1. *Assume that $u \in L^q(\mathbf{R}^n)$, where $q > \max\{\beta, n(\beta-1)/\alpha\}$ is a positive constant, is a positive solution of (2). Then it must be radially symmetric and monotone decreasing about some point.*

2. Bessel Potentials

For the convenience of readers, we recall some basic properties of Bessel potentials.

The Bessel kernel, g_α, $\alpha > 0$, is defined by

$$g_\alpha(x) = \frac{1}{\gamma(\alpha)} \int_0^\infty \exp(-\pi|x|^2/\delta)\exp(-\delta/4\pi)\delta^{(-n+\alpha)/2}\frac{d\delta}{\delta},$$

where $\gamma(\alpha) = (4\pi)^{\alpha/2}\Gamma(\alpha/2)$.

We now state, without proof, certain elementary facts about g_α. For details, one may refer to [10] or [12].

Proposition 2.

(1) The Fourier transform of g_α is

$$\widehat{g_\alpha}(x) = \frac{1}{(1+4\pi^2|x|^2)^{\alpha/2}}$$

where the Fourier transform is

$$\widehat{f}(x) = \int_{\mathbf{R}^n} f(t)\exp(-2\pi i x\cdot t)dt.$$

(2) For each $\alpha > 0$, $g_\alpha(x) \in L^1(\mathbf{R}^n)$.

(3) The following Bessel composition formula holds

$$g_\alpha * g_\beta = g_{\alpha+\beta}, \alpha, \beta \geq 0.$$

For any $\alpha \geq 0$, and $f \in L^p(\mathbf{R}^n)$, $1 \leq p \leq \infty$, we define the Bessel potentials $B_\alpha(f)$ as

$$B_\alpha(f) = g_\alpha * f$$

if $\alpha > 0$, and

$$B_0(f) = f.$$

Proposition 3.

(1) $\|B_\alpha(f)\|_{L^p(\mathbf{R}^n)} \leq \|f\|_{L^p(\mathbf{R}^n)}$, $1 \leq p \leq \infty$.

(2) $B_\alpha \cdot B_\beta = B_{\alpha+\beta}, \alpha, \beta \geq 0$.

Proof.

(1) follows from the Young's inequality and the fact that $\|g_\alpha\|_{L^1(\mathbf{R}^n)} = 1 (= \int_{\mathbf{R}^n} g_\alpha(x)dx)$.

(2) follows from the Bessel composition formula. □

We remark that on Sobolev spaces [10],

$$B_\alpha = (I - \triangle)^{-\alpha/2}.$$

The most interesting fact concerning Bessel potentials is that they can be employed to characterize the Sobolev spaces $W^{k,p}(\mathbf{R}^n)$. This is expressed in the following theorem where we employ the notation

$$L^{\alpha,p}(\mathbf{R}^n), \alpha > 0, 1 \leq p \leq \infty,$$

to denote all functions u such that

$$u = g_\alpha * f$$

for some $f \in L^p(\mathbf{R}^n)$. We define

$$\|u\|_{L^{\alpha,p}(\mathbf{R}^n)} = \|f\|_{L^p(\mathbf{R}^n)}.$$

With respect to this norm, $L^{\alpha,p}(\mathbf{R}^n)$ is a Banach space.

Theorem 4. *If k is a nonnegative integer and $1 < p < \infty$, then*

$$L^{k,p}(\mathbf{R}^n) = W^{k,p}(\mathbf{R}^n).$$

Moreover, if $u \in L^{k,p}(\mathbf{R}^n)$ with $u = g_\alpha * f$, then

$$C^{-1}\|f\|_{L^p(\mathbf{R}^n)} \leq \|u\|_{W^{k,p}(\mathbf{R}^n)} \leq C\|f\|_{L^p(\mathbf{R}^n)}$$

where $C = C(\alpha, n, p)$.

3. Imbedding Theorems

In this section, we prove some imbedding results which will play an important role in the proof of Theorem 1. First of all, let us recall the classical Sobolev imbedding theorem of the spaces $W^{k,p}(\mathbf{R}^n)$. For the proof, see, for instance, [1].

Theorem 5. *(The Sobolev imbedding theorem) For $1 \leq p < \infty$, there exist the following imbeddings:*

$$W^{k,p}(\mathbf{R}^n) \longrightarrow \begin{cases} L^q(\mathbf{R}^n), & kp < n, \text{ and } p \leq q \leq \frac{np}{n-kp}, \\ L^q(\mathbf{R}^n), & kp = n, \text{ and } p \leq q < \infty, \\ C_b(\mathbf{R}^n), & kp > n, \end{cases}$$

where $C_b(\mathbf{R}^n) = \{u \in C(\mathbf{R}^n) : u \text{ is bounded on } \mathbf{R}^n\}$.

The following lemma comes from [1](Theorem 7.63 (d) and (e)).

Lemma 6.

(1). *If* $t \le s$ *and* $1 < p \le q \le \frac{np}{n-(s-t)p} < \infty$, *then*

$$L^{s,p}(\mathbf{R}^n) \to L^{t,q}(\mathbf{R}^n).$$

(2). *If* $0 \le \mu \le s - n/p < 1$, *then*

$$L^{s,p}(\mathbf{R}^n) \to C^{0,\mu}(\mathbf{R}^n).$$

Now we can prove

Lemma 7. *Assume that* $q > \max\{\beta, n(\beta-1)/\alpha\}$. *If* $f \in L^{q/\beta}(\mathbf{R}^n)$, *then* $B_\alpha(f) \in L^q(\mathbf{R}^n)$. *Moreover, we have*

$$\|B_\alpha(f)\|_{L^q(\mathbf{R}^n)} \le C\|f\|_{L^{q/\beta}(\mathbf{R}^n)}$$

where $C = C(\alpha, \beta, n, q)$.

Proof. Obviously, we have that $B_\alpha(f) \in L^{\alpha,q/\beta}(\mathbf{R}^n)$ since $f \in L^{q/\beta}(\mathbf{R}^n)$. When α is an integer, by Theorem 4, $B_\alpha(f) \in W^{\alpha,q/\beta}(\mathbf{R}^n)$. Then we can directly use Theorem 5 to obtain the results. We leave the proof as an exercise to the interested readers. Here, we only give the proof of the case when α is a fraction, i.e., $\alpha - [\alpha] > 0$, where $[\alpha]$ denotes the integer satisfying $[\alpha] \le \alpha < [\alpha] + 1$. There are two cases.

Case 1. $q < \frac{n\beta}{\alpha - [\alpha]}$

Then by Lemma 6 (1), we have

$$B_\alpha(f) \in L^{[\alpha],r}(\mathbf{R}^n),$$

where $1 < q/\beta \le r \le r_0 < \infty$ and $r_0 = \frac{nq/\beta}{n-(\alpha-[\alpha])q/\beta}$. Since $[\alpha]$ is an integer, we have

$$B_\alpha(f) \in W^{[\alpha],r}(\mathbf{R}^n),$$

There are two subcases.

Case 1.1. $q \le r_0$

This is equivalent to

$$q \ge \frac{n(\beta-1)}{\alpha - [\alpha]}.$$

In this case, we have that $B_\alpha(f) \in W^{[\alpha],q}(\mathbf{R}^n)$. Moreover,

$$\|B_\alpha(f)\|_{L^q(\mathbf{R}^n)} \le C\|B_\alpha(f)\|_{L^{q/\beta}(\mathbf{R}^n)} = C\|f\|_{L^{q/\beta}(\mathbf{R}^n)}.$$

Case 1.2. $q > r_0$

This is equivalent to

$$q < \frac{n(\beta-1)}{\alpha - [\alpha]}.$$

In this case, we need to use Theorem 5 to raise the exponent.

When $q < n\beta/\alpha$, we have

$$[\alpha]r_0 < n.$$

To make sure that q is in $[r_0, \frac{nr_0}{n-[\alpha]r_0}]$, it requires that

$$q \ge n(\beta-1)/\alpha.$$

Then by Theorem 5, we get

$$B_\alpha(f) \in L^q(\mathbf{R}^n).$$

When $q = n\beta/\alpha$, we have

$$[\alpha] r_0 = n.$$

Then by Theorem 5, we get

$$B_\alpha(f) \in L^q(\mathbf{R}^n).$$

When $q > n\beta/\alpha$, we have

$$[\alpha] r_0 > n.$$

Then by Theorem 5, we get

$$B_\alpha(f) \in C_b(\mathbf{R}^n).$$

A straightforward calculation shows that

$$\begin{aligned}
\|B_\alpha(f)\|_{L^q(\mathbf{R}^n)} &\le \|B_\alpha(f)\|_{C_b(\mathbf{R}^n)}^{1-\frac{1}{\beta}} \|B_\alpha(f)\|_{L^{q/\beta}(\mathbf{R}^n)}^{\frac{1}{\beta}} \\
&\le C\|B_\alpha(f)\|_{L^{q/\beta}(\mathbf{R}^n)}^{1-\frac{1}{\beta}} \|B_\alpha(f)\|_{L^{q/\beta}(\mathbf{R}^n)}^{\frac{1}{\beta}} \\
&= C\|B_\alpha(f)\|_{L^{q/\beta}(\mathbf{R}^n)} \\
&= C\|f\|_{L^{q/\beta}(\mathbf{R}^n)}.
\end{aligned}$$

Case 2. $q \ge \frac{n\beta}{\alpha - [\alpha]}$

First, we see that

$$B_{[\alpha]}(f) \in L^{\alpha-[\alpha], q/\beta}(\mathbf{R}^n).$$

Notice that

$$0 \le \alpha - [\alpha] - \frac{n}{q/\beta} < 1 \Leftrightarrow q \ge \frac{n\beta}{\alpha - [\alpha]}.$$

So by Lemma 6 (2), we have

$$B_{[\alpha]}(f) \in C^{0,\mu}(\mathbf{R}^n),$$

where $0 \le \mu \le \alpha - [\alpha] - \frac{n}{q/\beta}$. Then as before, we compute

$$\begin{aligned}
\|B_{[\alpha]}(f)\|_{L^q(\mathbf{R}^n)} &\le \|B_{[\alpha]}(f)\|_{C^{0,\mu}(\mathbf{R}^n)}^{1-\frac{1}{\beta}} \|B_{[\alpha]}(f)\|_{L^{q/\beta}(\mathbf{R}^n)}^{\frac{1}{\beta}} \\
&\le C\|B_{[\alpha]}(f)\|_{L^{q/\beta}(\mathbf{R}^n)}^{1-\frac{1}{\beta}} \|B_{[\alpha]}(f)\|_{L^{q/\beta}(\mathbf{R}^n)}^{\frac{1}{\beta}} \\
&= C\|B_{[\alpha]}(f)\|_{L^{q/\beta}(\mathbf{R}^n)} \\
&= C\|f\|_{L^{q/\beta}(\mathbf{R}^n)}.
\end{aligned}$$

By now, we have finished the proof. $\square$

4. Proof of Theorem 1

For a given real number λ, define

$$\Sigma_\lambda = \{x = (x_1, \cdots, x_n) | x_1 \geq \lambda\}.$$

Let $x^\lambda = (2\lambda - x_1, \cdots, x_n)$, $u_\lambda(x) = u(x^\lambda)$.

Lemma 8. *For any solution $u(x)$ of (2), we have*

$$u(x) - u_\lambda(x) = \int_{\Sigma_\lambda} (g_\alpha(x-y) - g_\alpha(x^\lambda - y))(u(y)^\beta - u_\lambda(y)^\beta) dy.$$

Proof. Let

$$\Sigma_\lambda^c = \{x = (x_1, \cdots, x_n) | x_1 < \lambda\}.$$

Then it is easy to see that

$$\begin{aligned} u(x) &= \int_{\Sigma_\lambda} g_\alpha(x-y) u(y)^\beta dy + \int_{\Sigma_\lambda^c} g_\alpha(x-y) u(y)^\beta dy \\ &= \int_{\Sigma_\lambda} g_\alpha(x-y) u(y)^\beta dy + \int_{\Sigma_\lambda} g_\alpha(x^\lambda - y) u_\lambda(y)^\beta dy. \end{aligned}$$

Substituting x by x^λ, we get

$$u(x^\lambda) = \int_{\Sigma_\lambda} g_\alpha(x^\lambda - y) u(y)^\beta dy + \int_{\Sigma_\lambda} g_\alpha(x-y) u_\lambda(y)^\beta dy.$$

Thus

$$\begin{aligned} & u(x) - u(x^\lambda) \\ = & \int_{\Sigma_\lambda} g_\alpha(x-y)(u(y)^\beta - u_\lambda(y)^\beta) dy - \int_{\Sigma_\lambda} g_\alpha(x^\lambda - y)(u(y)^\beta - u_\lambda(y)^\beta) dy \\ = & \int_{\Sigma_\lambda} (g_\alpha(x-y) - g_\alpha(x^\lambda - y))(u(y)^\beta - u_\lambda(y)^\beta) dy. \end{aligned}$$

□

Proof of Theorem 1. Our proof is divided into two steps.

Step 1. Define

$$\Sigma_\lambda^- = \{x | x \in \Sigma_\lambda, u(x) < u_\lambda(x)\}.$$

We want to show that for sufficiently negative values of λ, Σ_λ^- must be empty.

Whenever $x, y \in \Sigma_\lambda$, we have that $|x-y| \leq |x^\lambda - y|$. Then by the definition of g_α, we have

$$g_\alpha(x-y) \geq g_\alpha(x^\lambda - y).$$

Then by Lemma 8, for $x \in \Sigma_\lambda^-$,

$$\begin{aligned} u_\lambda(x) - u(x) &\leq \int_{\Sigma_\lambda^-} (g_\alpha(x-y) - g_\alpha(x^\lambda - y))(u_\lambda(y)^\beta - u(y)^\beta) dy \\ &\leq \beta \int_{\Sigma_\lambda^-} g_\alpha(x-y)[u_\lambda^{\beta-1}(u_\lambda - u)](y) dy. \end{aligned}$$

It follows first from Lemma 7 and then the *Hölder* inequality that

$$
\begin{aligned}
\|u_\lambda - u\|_{L^q(\Sigma_\lambda^-)} &\le \beta \|B_\alpha(u_\lambda^{\beta-1}(u_\lambda - u))\|_{L^q(\Sigma_\lambda^-)} \\
&\le C\|u_\lambda^{\beta-1}(u_\lambda - u)\|_{L^{q/\beta}(\Sigma_\lambda^-)} \\
&\le C(\int_{\Sigma_\lambda^-} u_\lambda(y)^q dy)^{\frac{\beta-1}{q}} \|u_\lambda - u\|_{L^q(\Sigma_\lambda^-)} \\
&\le C(\int_{\Sigma_\lambda^c} u(y)^q dy)^{\frac{\beta-1}{q}} \|u_\lambda - u\|_{L^q(\Sigma_\lambda^-)} \qquad (3)
\end{aligned}
$$

Since $u \in L^q(\mathbf{R}^n)$, we can choose N sufficiently large, such that for $\lambda \le -N$, we have

$$
C(\int_{\Sigma_\lambda^c} u(y)^q dy)^{\frac{\beta-1}{q}} \le \frac{1}{2}.
$$

Now (3) implies that

$$
\|u_\lambda - u\|_{L^q(\Sigma_\lambda^-)} = 0,
$$

and therefore Σ_λ^- must be measure zero, and hence empty.

Step 2. Now we have that for $\lambda \le -N$,

$$
u(x) \ge u_\lambda(x), \quad \forall x \in \Sigma_\lambda. \tag{4}
$$

Thus we can start moving the plane continuously from $\lambda \le -N$ to the right as long as (4) holds. Suppose that at a $\lambda_0 < 0$, we have $u(x) \ge u_{\lambda_0}(x)$, but *meas* $\{x \in \Sigma_{\lambda_0} | u(x) > u_{\lambda_0}(x)\} > 0$. We will show that the plane can be moved further to the right, i.e., there exists an ϵ depending on n, α, β, q and the solution u such that $u(x) \ge u_\lambda(x)$ on Σ_λ for all λ in $[\lambda_0, \lambda_0 + \epsilon)$.

By Lemma 8, we see that $u(x) > u_{\lambda_0}(x)$ in the interior of Σ_{λ_0}. Let

$$
\overline{\Sigma_{\lambda_0}^-} = \{x \in \Sigma_{\lambda_0} | u(x) \le u_{\lambda_0}(x)\}.
$$

Then it is easy to see that $\overline{\Sigma_{\lambda_0}^-}$ has measure zero, and $\lim_{\lambda \to \lambda_0} \Sigma_\lambda^- \subset \overline{\Sigma_{\lambda_0}^-}$. Let $(\Sigma_\lambda^-)^\star$ be the reflection of Σ_λ^- about the plane $x_1 = \lambda$. From the third inequality of (3), we get

$$
\|u_\lambda - u\|_{L^q(\Sigma_\lambda^-)} \le C(\int_{(\Sigma_\lambda^-)^\star} u(y)^q dy)^{\frac{\beta-1}{q}} \|u_\lambda - u\|_{L^q(\Sigma_\lambda^-)}. \tag{5}
$$

Since $u \in L^q(\mathbf{R}^n)$, then one can choose ϵ small enough, such that for all λ in $[\lambda_0, \lambda_0 + \epsilon)$,

$$
C(\int_{(\Sigma_\lambda^-)^\star} u(y)^q dy)^{\frac{\beta-1}{q}} \le \frac{1}{2}.
$$

Now by (5), we have

$$
\|u_\lambda - u\|_{L^q(\Sigma_\lambda^-)} = 0,
$$

and therefore Σ_λ^- must be empty.

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